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Self-similar motion of three point vortices

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One of the counter-intuitive results in the three-vortex problem is that the vortices can converge on and meet at a point in a finite time for certain sets of vortex circulations and for certain initial conditions. This result was already included in Gröbli's thesis of 1877 and has since been elaborated by several authors. It arises from an investigation of motions where the vortex triangle retains its shape for all time, but not its size. We revisit these self-similar motions, develop a new derivation of the initial conditions that lead to them, and derive a number of formulae pertaining to the rate of expansion or collapse and the angular frequency of rotation, some of which appear to be new. We also pursue the problem of linear stability of these motions in detail and, again, provide a number of formulae, some of which are new. In particular, we determine all eigenmodes analytically.

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I. INTRODUCTION

The point vortex equations on the unbounded plane, taken as the complex plane, without background flow are

$$\dot{\bar{z}}_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^N \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad (1)$$

where the dot signifies a time derivative, the line over a symbol means complex conjugation, and the prime on the summation means $\beta \neq \alpha$. The time-dependent vortex positions have been denoted $z_\alpha(t)$, $\alpha=1, \dots, N$. The constant circulations of the vortices are Γ_α . These equations may be found in many places in the literature.¹⁻³

The general theory of point vortex dynamics is extensive, and we shall only list what we need here. From Eq. (1) one may derive⁴⁻⁶ equations of motion for the intervortex separations $s_1=|z_2-z_3|$, $s_2=|z_3-z_1|$, and $s_3=|z_1-z_2|$, thus

$$\begin{aligned} \frac{ds_1^2}{dt} &= \frac{2\Gamma_1}{\pi} \Delta \frac{s_3^2 - s_2^2}{s_2^2 s_3^2}, \\ \frac{ds_2^2}{dt} &= \frac{2\Gamma_2}{\pi} \Delta \frac{s_1^2 - s_3^2}{s_3^2 s_1^2}, \\ \frac{ds_3^2}{dt} &= \frac{2\Gamma_3}{\pi} \Delta \frac{s_2^2 - s_1^2}{s_1^2 s_2^2}, \end{aligned} \quad (2)$$

where Δ is given up to its sign by Heron's formula for the area of a triangle in terms of its sides:

$$16\Delta^2 = 2s_2^2 s_3^2 + 2s_3^2 s_1^2 + 2s_1^2 s_2^2 - s_1^4 - s_2^4 - s_3^4. \quad (3a)$$

Care must be taken with the sign of Δ in Eq. (2). Gröbli⁴ finds

$$\Delta = \frac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_3 y_2 - x_2 y_1). \quad (3b)$$

This makes $\Delta > 0$ when vortices 123 appear counterclockwise, and negative when they appear clockwise. The magnitude of Δ is given by Eq. (3b) for either orientation, so the system (2) is "closed" in the three variables s_1, s_2, s_3 except for instants when the vortices become collinear.

Equations (1) have four known general integrals. They are (i) the two (real) components, X and Y , of the linear impulse

$$X + iY = \sum_{\alpha=1}^N \Gamma_\alpha z_\alpha;$$

(ii) the angular impulse,

$$I = \sum_{\alpha=1}^N \Gamma_\alpha |z_\alpha|^2;$$

and (iii) the Hamiltonian of the point vortex system

$$H = -\frac{1}{2\pi} \sum_{1 \leq \alpha < \beta \leq N} \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|.$$

The combination,

$$L = \sum_{\alpha=1}^N \Gamma_\alpha I - X^2 - Y^2,$$

is extremely useful since it only involves the vortex separations. For three vortices

$$L = \Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 + \Gamma_1 \Gamma_2 s_3^2. \quad (4a)$$

The Hamiltonian for three vortices is

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$$H = -\frac{1}{2\pi}(\Gamma_2\Gamma_3 \log s_1 + \Gamma_3\Gamma_1 \log s_2 + \Gamma_1\Gamma_2 \log s_3). \quad (4b)$$

It is not difficult to verify that L and H are, indeed, conserved by Eqs. (2).

We shall also find the following notation convenient. The three symmetric functions of the vortex strengths are designated

$$\gamma_1 = \Gamma_1 + \Gamma_2 + \Gamma_3, \quad (5a)$$

$$\gamma_2 = \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1, \quad (5b)$$

$$\gamma_3 = \Gamma_1\Gamma_2\Gamma_3. \quad (5c)$$

We shall say that the motion of N point vortices is *self-similar* if the vortex configuration remains of the same shape for all time although the size of the configuration may change. The relative equilibria of such a system may, of course, be thought of as a special case of self-similar motion when, in fact, both the shape and the size of the configuration remain invariant. Self-similar motions of point vortices are, indeed, closely related to relative equilibria.

Self-similar motions of three point vortices were conceived, identified and explored by Gröbli.⁴ Later Synge⁷ studied them in some detail as part of a general classification of solutions to the three-vortex problem. These solutions were rediscovered independently by Aref⁶ and studied by Novikov and Sedov.⁸ They have subsequently been studied by several authors. Of particular importance to the present paper are the analyses by Tavantzis and Ting⁹ and Kimura.¹⁰

For a given triple of circulations consider an initial condition that leads to self-similar expansion of a vortex configuration (we shall determine necessary and sufficient conditions momentarily). Now, consider the same configuration reflected in one of the sides of the triangle. The time evolution of the vortex separations in the second system will be the exact reverse of the first system, since the reflection will flip the sign of Δ while retaining the values of s_1, s_2, s_3 . Thus, if we can find self-similar motions that are not relative equilibria for three vortices with given circulations, we must be able to find both expanding and collapsing configurations. If the collapse is sufficiently rapid, and we shall see below that it is, we arrive at the notion of a *singularity after a finite time* in the point vortex equations. This is the phenomenon of *vortex collapse*.

Vortex collapse, of course, contradicts the intuitive notion that “the distance between any two vortices can never be much less than the smallest distance between any pair of vortices initially.”¹ We shall see, however, that the set of vortex strengths and the initial conditions for which this type of motion arises are, indeed, quite limited. The feature of the N -vortex problem that it has a “singularity after a finite time” for certain initial conditions is in sharp contrast with the two-dimensional (2D) Euler equation with smooth initial data, which may be shown to remain regular for all time. This result is due to a classical paper by Wolibner¹¹ and has its own mathematical literature that we shall not review. The reader desiring a starting point in English may find the paper

by Schaeffer¹² useful. The difference is not terribly surprising since the vorticity distribution in the point vortex problem is, of course, singular when viewed as a 2D vorticity field. It is, nevertheless, true that arbitrarily smooth solutions of the 2D Euler equation, representing three concentrated vortices, may be found such that the solution tracks the collapse solution for three point vortices for a very long time. Only as the three concentrated vortices approach one another very closely do they “discover” that they cannot collide as in three-vortex collapse, and a rapid, in some cases quite dramatic, departure from the three-vortex solution takes place. The study by Vosbeek *et al.*¹³ is a good starting point for this set of issues.

If the point vortices are thought of as a set of rectilinear vortices in three dimensions, the 2D collapse solutions become collapse solutions in three-dimensional (3D) albeit with vorticity extending to infinity. Even if the vortices are made to “close up” at large distances, one would still expect the rectilinear portions to execute a near-collapse motion for a considerable interval of time before possibly deviating from it. Such considerations motivated searches for 3D vortex collapse models that attempt to address finite-time singularity formation in the 3D Euler equation.¹⁴

II. DETERMINATION OF SELF-SIMILAR MOTIONS

We seek motions for which $s_\alpha(t) = f(t)s_\alpha(0)$ with the same function $f(t)$ for $\alpha=1,2,3$. Substituting this into the first of Eqs. (2), we find

$$\frac{ds_1^2}{dt} = \frac{df^2}{dt}s_1^2(0) = 2\frac{\Gamma_1}{\pi}\Delta(0)\frac{s_3^2(0)-s_2^2(0)}{s_2^2(0)s_3^2(0)}. \quad (6a)$$

The right hand side in Eq. (6a) is a constant, so f^2 must be linear in t . For consistency the constants obtained from the first, second, and third of Eqs. (2) must give the same rate of change of f^2 , i.e., we must require that

$$\Gamma_1(s_3^2 - s_2^2) = \Gamma_2(s_1^2 - s_3^2) = \Gamma_3(s_2^2 - s_1^2). \quad (6b)$$

We shall soon understand condition (6b) in terms of the conservation of L , Eq. (4a). In any event, our first necessary condition for self-similar motion, is that

$$f(t) = \sqrt{1 - \frac{t}{\tau}}, \quad (7)$$

where the aforementioned constant rate of change of f^2 has been called $-1/\tau$, and we used the initial condition $f(0)=1$. The physical interpretation of τ is that its magnitude sets a time scale for the relative motion. If $\tau > 0$, it may be interpreted as the *collapse time*. In particular, we note that the necessary condition gives a finite time to collapse.

During self-similar motion the integrals H and L , Eqs. (4a) and (4b) respectively, must be conserved as for any motion. With the intervortex distances changing as $f(t)$, we see that H will evolve according to

$$H(t) = H(0) - \frac{\gamma_2}{2\pi} \log f(t),$$

where γ_2 was given by Eq. (5b). Thus, as a further necessary condition for self-similar motion, the three vortex circulations must satisfy $\gamma_2=0$.

Further, since L will be multiplied by $f^2(t)$ during the motion, a second necessary condition is that the initial positions of the vortices satisfy $L=0$. The necessary conditions $\gamma_2=0$ and $L=0$ turn out also to be sufficient to determine the self-similar motions of three vortices. We first note that the right hand sides in Eq. (6a), and its variants for s_2 and s_3 determining $-1/\tau$ will, indeed, all have the same value when $\gamma_2=0$ and $L=0$. This follows by using $\gamma_2=0$ to write

$$L = \Gamma_1 \Gamma_2 s_3^2 + \Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 = 0$$

as

$$-(\Gamma_2 \Gamma_3 + \Gamma_3 \Gamma_1) s_3^2 + \Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 = 0,$$

from which one easily deduces Eq. (6b). Thus, if $L=0$ for three vortices with strengths that satisfy $\gamma_2=0$, we have

$$\frac{1}{s_1^2} \frac{ds_1^2}{dt} = \frac{1}{s_2^2} \frac{ds_2^2}{dt} = \frac{1}{s_3^2} \frac{ds_3^2}{dt}.$$

We may set the common value of these three relative rates of change to $2(\log f)'$, and we have established that the motion is self-similar. The conditions $\gamma_2=0$ and $L=0$ are, thus, both necessary and sufficient for three vortices to evolve in a self-similar motion.

A. Initial conditions for self-similar motion

Assuming $\gamma_2=0$, we can now explicitly construct initial conditions for three vortices with $L=0$. Consider a system of coordinates with the x -axis chosen as the line through vortices 1 and 2, and the origin at the center of vorticity of these two vortices. Thus, the y -axis is chosen such that vortex 1 is at $x_1 = -\Gamma_2 d / (\Gamma_1 + \Gamma_2)$, and vortex 2 is at $x_2 = \Gamma_1 d / (\Gamma_1 + \Gamma_2)$. Here d is the distance between vortices 1 and 2. In order to have $L=0$, the coordinates, (x_3, y_3) , of vortex 3 must satisfy

$$\Gamma_1 \Gamma_2 d^2 + \Gamma_2 \Gamma_3 [(x_2 - x_3)^2 + y_3^2] + \Gamma_3 \Gamma_1 [(x_1 - x_3)^2 + y_3^2] = 0,$$

or

$$\Gamma_1 \Gamma_2 d^2 + \Gamma_3 (\Gamma_1 x_1^2 + \Gamma_2 x_2^2) - 2\Gamma_3 (\Gamma_1 x_1 + \Gamma_2 x_2) x_3 + \Gamma_3 (\Gamma_1 + \Gamma_2) (x_3^2 + y_3^2) = 0,$$

or, finally,

$$x_3^2 + y_3^2 = \frac{\Gamma_1 + \Gamma_2 + \Gamma_3}{\Gamma_1 + \Gamma_2} d^2. \quad (8)$$

This is the equation of a circle, shown in Fig. 1, centered at the origin, which is also the center of vorticity of vortices 1 and 2, with a radius that is the distance between these two vortices multiplied by $\sqrt{\gamma_1 / (\Gamma_1 + \Gamma_2)}$. This geometric construction was first presented by Aref.⁶ Kimura provides further discussion.¹⁵

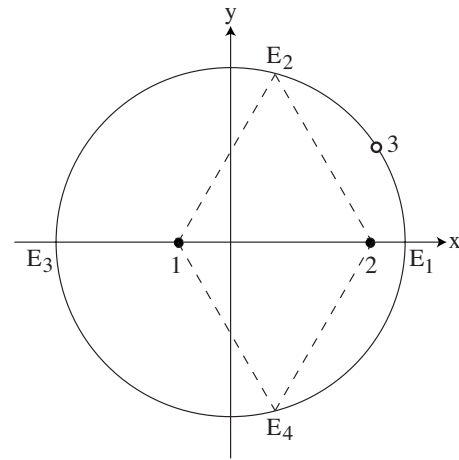


FIG. 1. Construction of initial conditions leading to self-similar collapse/expansion for three vortices with $\gamma_2=0$.

The bisector of the segment connecting 1 and 2 is the line

$$x_3 = \frac{1}{2}(x_1 + x_2) = \frac{\Gamma_1 - \Gamma_2 d}{\Gamma_1 + \Gamma_2}.$$

It intersects the circle at $y_3 = \pm d(\sqrt{3}/2)$, i.e., at the two points on the circle where triangle 123 is equilateral. Thus, the circle (8) may be constructed knowing just the center of vorticity of vortices 1 and 2 and the distance between them. The equilateral triangle based on the positions of vortices 1 and 2 provides a point on the circle and, hence, its radius. The x -axis intersects the circle in the points E_1 and E_3 of Fig. 1 with coordinates $(\pm d\sqrt{\gamma_1 / (\Gamma_1 + \Gamma_2)}, 0)$, respectively. These points correspond to the two rotating, collinear relative equilibria discussed in Ref. 16 and in what follows. The points E_2 and E_4 clearly correspond to equilateral triangle relative equilibria.

Except for the four points, indicated as E_1 , E_2 , E_3 , and E_4 in Fig. 1, any initial condition with vortex 3 on the circle will lead to self-similar collapse or expansion of the vortex triangle with time. The sign of τ is best determined from the equation

$$\frac{ds_3^2}{dt} = 2 \frac{\Gamma_3}{\pi} \Delta(0) \frac{s_2^2(0) - s_1^2(0)}{s_1^2(0)s_2^2(0)} = -\frac{s_3^2(0)}{\tau}.$$

For vortex 3 on the arc $E_1 E_2$, we have $s_2 > s_1$ and, since vortices 123 appear counterclockwise, $\Delta > 0$. Since $\Gamma_3 < 0$, we have $\tau > 0$ and the vortex triangle will collapse. On the arc $E_2 E_3$ we have $s_2 < s_1$ but Δ is still positive. Thus, $\tau < 0$ and the vortex triangle expands self-similarly. We get self-similar collapse for vortex 3 on the arc $E_3 E_4$, self-similar expansion on $E_4 E_1$.

Consider perturbations of the relative equilibria in which we displace vortex 3 infinitesimally along the circle in Fig. 1. The integral L retains its value 0 by construction. Since each relative equilibrium configuration sits at the crossover between expansion and collapse configurations, the perturbed configuration will either expand or collapse depending on which way vortex 3 is displaced along the circle. This explains qualitatively why in a linear perturbation analysis the

four relative equilibria for $\gamma_2=0$ that are “adjacent” to self-similar motions show up as marginally stable.

B. Central configurations

Gröbli⁴ showed that the ODEs for the polar coordinates of three vortices can be written in a form where, apart from the first derivative of any given polar coordinate, only the sides of the vortex triangle appear. If the self-similar evolution of the sides is substituted in his formulae, they lead to the complex positions of the vortices evolving as

$$z_\alpha(t) = z_\alpha(0) \left(1 - \frac{t}{\tau}\right)^{(1/2) - i\Omega\tau}, \quad (9)$$

where τ has already been introduced, and Ω is a second real parameter. The signs have been set such that both τ and Ω are positive when the configuration collapses as it rotates in the counterclockwise direction about the center of vorticity. In the limit $\tau \rightarrow \infty$ we obtain steadily rotating configurations, i.e., relative equilibria.

We may, thus, approach the problem of self-similar motion by asking that the vortices move so that the configuration rotates—translation is ruled out since the sum of circulations is nonzero—while changing scale. We are then led to the *ansatz*

$$z_\alpha(t) = \lambda(t) z_\alpha(0). \quad (10a)$$

This *ansatz* corresponds to what we might call *central configurations* of three point vortices by analogy with celestial mechanics. Substitution into the point vortex equations now gives

$$\lambda \dot{\lambda} = \text{const.} \quad (10b)$$

If the constant, which is in general complex, is written as $-(1/2\tau) - i\Omega$, with two real parameters Ω and τ , and if we use the initial condition $\lambda(0)=1$, the solution to Eq. (10b) is

$$\lambda(t) = \left(1 - \frac{t}{\tau}\right)^{(1/2) - i\Omega\tau}, \quad (11)$$

i.e., we obtain Eq. (9).

We easily get the following necessary conditions on the known integrals for such configurations:

$$\sum_{\alpha=1}^N \Gamma_\alpha z_\alpha = 0, \quad \sum_{\alpha=1}^N \Gamma_\alpha |z_\alpha|^2 = 0, \quad (12)$$

along with the condition $\gamma_2=0$ from conservation of the Hamiltonian. Combining the two conditions Eq. (12) gives $L=0$.

The equations to be satisfied by the initial positions of the vortices are, cf. Eq. (1),

$$\left(-\frac{1}{2\tau} - i\Omega\right) \bar{z}_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^N \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \quad (13)$$

Note that if $\{z_1, z_2, z_3\}$ is a solution to these equations for certain values of Ω and τ , then the reflected configuration, $\{\bar{z}_1, \bar{z}_2, \bar{z}_3\}$, is also a solution with parameters Ω and $-\tau$. From Eq. (13) we have the condition that

$$\frac{1}{\bar{z}_\alpha} \sum_{\beta=1}^N \frac{\Gamma_\beta}{z_\alpha - z_\beta}$$

be independent of α . For three vortices this gives

$$\begin{aligned} \frac{1}{\bar{z}_1} \left[\frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3} \right] &= \frac{1}{\bar{z}_2} \left[\frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3} \right] \\ &= \frac{1}{\bar{z}_3} \left[\frac{\Gamma_1}{z_3 - z_1} + \frac{\Gamma_2}{z_3 - z_2} \right]. \end{aligned}$$

Comparing any two of these, and using $\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = 0$, gives an analog of “Gröbli’s relation” for relative equilibria, cf. Eq. (28b) of Ref. 16:

$$\frac{\bar{z}_1}{z_2 - z_3} + \frac{\bar{z}_2}{z_3 - z_1} + \frac{\bar{z}_3}{z_1 - z_2} = 0. \quad (14)$$

We can now proceed as we did for the collinear relative equilibria¹⁶ by introducing variables

$$\zeta_1 = z_2 - z_3, \quad \zeta_2 = z_3 - z_1, \quad \zeta_3 = z_1 - z_2, \quad (15a)$$

which are, once again, not independent but sum to 0. We then have

$$\frac{\bar{z}_1}{\zeta_1} + \frac{\bar{z}_2}{\zeta_2} + \frac{\bar{z}_3}{\zeta_3} = 0, \quad (15b)$$

and the representation inverse to Eq. (15a):

$$z_1 = \frac{\Gamma_2 \zeta_3 - \Gamma_3 \zeta_2}{\gamma_1}, \quad (16a)$$

$$z_2 = \frac{\Gamma_3 \zeta_1 - \Gamma_1 \zeta_3}{\gamma_1}, \quad (16b)$$

$$z_3 = \frac{\Gamma_1 \zeta_2 - \Gamma_2 \zeta_1}{\gamma_1}. \quad (16c)$$

Since $\gamma_2=0$ we are assured of the denominator $\gamma_1 \neq 0$ in these formulae. Equation (15b) now becomes

$$\Gamma_1 \left(\frac{\bar{\zeta}_2}{\zeta_3} - \frac{\bar{\zeta}_3}{\zeta_2} \right) + \Gamma_2 \left(\frac{\bar{\zeta}_3}{\zeta_1} - \frac{\bar{\zeta}_1}{\zeta_3} \right) + \Gamma_3 \left(\frac{\bar{\zeta}_1}{\zeta_2} - \frac{\bar{\zeta}_2}{\zeta_1} \right) = 0.$$

If we set

$$z = \frac{\bar{\zeta}_1}{\zeta_3} \quad (17)$$

and assume the coordinates rotated such that ζ_3 is real, we have

$$\frac{\bar{\zeta}_3}{\zeta_1} = \frac{1}{\bar{z}}, \quad \frac{\bar{\zeta}_2}{\zeta_3} = -z - 1, \quad \frac{\bar{\zeta}_1}{\zeta_2} = -\frac{1}{\bar{z} + 1},$$

$$\frac{\bar{\zeta}_1}{\zeta_2} = -\frac{z}{\bar{z} + 1}, \quad \frac{\bar{\zeta}_2}{\zeta_1} = -\frac{z + 1}{\bar{z}}.$$

Thus,

$$\Gamma_1 \left(-z - 1 + \frac{1}{1+\bar{z}} \right) + \Gamma_2 \left(\frac{1}{\bar{z}} - z \right) + \Gamma_3 \left(-\frac{z}{\bar{z}+1} + \frac{z+1}{\bar{z}} \right) = 0.$$

Multiplying through by $\bar{z}(\bar{z}+1)$ we get

$$\Gamma_1(1 - |z+1|^2)\bar{z} + \Gamma_2(1 - |z|^2)(\bar{z}+1) + \Gamma_3(|z+1|^2 - |z|^2) = 0. \quad (18)$$

Unless z is real, in which case we return to the collinear relative equilibria, the imaginary part of this equation gives

$$\Gamma_1(1 - |z+1|^2) + \Gamma_2(1 - |z|^2) = 0. \quad (19a)$$

There then remains

$$\Gamma_2(1 - |z|^2) + \Gamma_3(|z+1|^2 - |z|^2) = 0. \quad (19b)$$

Introducing the real and imaginary parts of z as $z=x+iy$, where x and y are real, we obtain from Eq. (19a)

$$(\Gamma_1 + \Gamma_2)(x^2 + y^2) + 2\Gamma_1 x = \Gamma_2,$$

and from Eq. (19b)

$$\Gamma_2(x^2 + y^2) - 2\Gamma_3 x = \Gamma_2 + \Gamma_3.$$

Since $\gamma_2=0$, these two equations are, in fact, the same equation (multiply the first by Γ_3 and the second by Γ_1), and may be written

$$\left(x + \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \right)^2 + y^2 = \left(\frac{\Gamma_*}{\Gamma_1 + \Gamma_2} \right)^2, \quad (20a)$$

where

$$\Gamma_* = \sqrt{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2}. \quad (20b)$$

This defines a circle in the (x,y) -plane, centered at $[-\Gamma_1/(\Gamma_1 + \Gamma_2), 0]$ with radius $\Gamma_*/(\Gamma_1 + \Gamma_2)$.

In our earlier treatment we had placed vortices 1 and 2 at

$$z_1 = -\frac{\Gamma_2 d}{\Gamma_1 + \Gamma_2}, \quad z_2 = \frac{\Gamma_1 d}{\Gamma_1 + \Gamma_2}, \quad (21a)$$

respectively. We then found that vortex 3 should be situated on the circle

$$z_3 = d \sqrt{\frac{\gamma_1}{\Gamma_1 + \Gamma_2}} e^{i\vartheta}, \quad 0 \leq \vartheta < 2\pi, \quad (21b)$$

in order to have self-similar motion. Using $\zeta_3 = -d$, and

$$\gamma_1 = \Gamma_1 + \Gamma_2 - \frac{\Gamma_1 \Gamma_2}{\Gamma_2 + \Gamma_2} = \frac{\Gamma_*^2}{\Gamma_1 + \Gamma_2},$$

which follows from $\gamma_2=0$, we get

$$\zeta_1 = z_2 - z_3 = \frac{\Gamma_1 - \Gamma_* e^{i\vartheta}}{\Gamma_1 + \Gamma_2} d, \quad (22)$$

$$z = \frac{-\Gamma_1 + \Gamma_* e^{-i\vartheta}}{\Gamma_1 + \Gamma_2},$$

i.e., that z orbits the circle (20a).

The point E_2 in Fig. 1 corresponds to $z_3 - z_2 = (z_2 - z_1)e^{i2\pi/3}$, or $\zeta_1 = \zeta_3 e^{i2\pi/3}$, or $z = e^{-i2\pi/3}$. The corresponding value of the angle ϑ in Eq. (21b) or Eq. (22) is ϑ_0 given by

$$\Gamma_* e^{-i\vartheta_0} = (1/2)(\Gamma_1 - \Gamma_2) - i(\sqrt{3}/2)(\Gamma_1 + \Gamma_2),$$

or

$$\cos \vartheta_0 = \frac{\Gamma_1 - \Gamma_2}{2\Gamma_*}, \quad \sin \vartheta_0 = \frac{\sqrt{3}\Gamma_1 + \Gamma_2}{2\Gamma_*}.$$

The point E_4 corresponds to the angle $-\vartheta_0$ as determined from these formulae.

To determine the parameters Ω and τ we return to Eq. (13) which gives the conditions for a configuration to lead to self-similar motion:

$$\left(2\pi\Omega - i\frac{\pi}{\tau} \right) \bar{z}_1 = \frac{\Gamma_2}{\zeta_3} - \frac{\Gamma_3}{\zeta_2},$$

$$\left(2\pi\Omega - i\frac{\pi}{\tau} \right) \bar{z}_2 = \frac{\Gamma_3}{\zeta_1} - \frac{\Gamma_1}{\zeta_3},$$

$$\left(2\pi\Omega - i\frac{\pi}{\tau} \right) \bar{z}_3 = \frac{\Gamma_1}{\zeta_2} - \frac{\Gamma_2}{\zeta_1}.$$

Subtracting the second of these from the first gives

$$\begin{aligned} \left(2\pi\Omega - i\frac{\pi}{\tau} \right) \bar{\zeta}_3 &= \frac{\Gamma_1 + \Gamma_2}{\zeta_3} - \Gamma_3 \left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2} \right) \\ &= \frac{\Gamma_1 + \Gamma_2}{\zeta_3} + \Gamma_3 \frac{\zeta_3}{\zeta_1 \zeta_2}, \end{aligned}$$

or

$$2\pi\Omega - i\frac{\pi}{\tau} - \frac{\Gamma_1 + \Gamma_2}{s_3^2} = \frac{\Gamma_3}{s_3^2} \frac{\zeta_3^2}{\zeta_1 \zeta_2}, \quad (23a)$$

and similarly

$$2\pi\Omega - i\frac{\pi}{\tau} - \frac{\Gamma_2 + \Gamma_3}{s_1^2} = \frac{\Gamma_1}{s_1^2} \frac{\zeta_1^2}{\zeta_2 \zeta_3}, \quad (23b)$$

$$2\pi\Omega - i\frac{\pi}{\tau} - \frac{\Gamma_3 + \Gamma_1}{s_2^2} = \frac{\Gamma_2}{s_2^2} \frac{\zeta_2^2}{\zeta_3 \zeta_1}. \quad (23c)$$

Multiplying Eqs. (23a)–(23c), we see that $2\pi\Omega - i(\pi/\tau)$ must be a root of the cubic equation

$$\left(Z - \frac{\Gamma_1 + \Gamma_2}{s_3^2} \right) \left(Z - \frac{\Gamma_2 + \Gamma_3}{s_1^2} \right) \left(Z - \frac{\Gamma_3 + \Gamma_1}{s_2^2} \right) = \frac{\Gamma_1 \Gamma_2 \Gamma_3}{s_1^2 s_2^2 s_3^2}. \quad (24)$$

This equation has the solution $Z=0$ since $(\Gamma_1 + \Gamma_2)(\Gamma_2 + \Gamma_3) \times (\Gamma_3 + \Gamma_1) = -\Gamma_1 \Gamma_2 \Gamma_3$, when $\gamma_2=0$. Indeed, one possible solution to Eq. (13) is that we have a stationary configuration, $\Omega=0$, $\tau=\infty$, i.e., the left hand sides of these equations are all 0. For such a configuration we must have $\Gamma_1 \zeta_1 = \Gamma_2 \zeta_2 = \Gamma_3 \zeta_3$. This configuration is the stationary collinear relative equilibrium that we studied in Ref. 16.

The polynomial in Eq. (24) has real coefficients. The remaining two roots must then be complex conjugates, i.e., $2\pi\Omega \mp i(\pi/\tau)$. Comparing the polynomial,

$$Z\left(Z - 2\pi\Omega - i\frac{\pi}{\tau}\right)\left(Z - 2\pi\Omega + i\frac{\pi}{\tau}\right),$$

to the polynomial in Eq. (24), we read off general relations for Ω and τ :

$$\Omega = \frac{1}{4\pi} \left[\frac{\Gamma_2 + \Gamma_3}{s_1^2} + \frac{\Gamma_3 + \Gamma_1}{s_2^2} + \frac{\Gamma_1 + \Gamma_2}{s_3^2} \right], \quad (25a)$$

$$(2\pi\Omega)^2 + \left(\frac{\pi}{\tau}\right)^2 = \frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2}. \quad (25b)$$

Now,

$$\begin{aligned} & \left[\frac{\Gamma_2 + \Gamma_3}{s_1^2} + \frac{\Gamma_3 + \Gamma_1}{s_2^2} + \frac{\Gamma_1 + \Gamma_2}{s_3^2} \right]^2 \\ &= \frac{(\Gamma_2 + \Gamma_3)^2}{s_1^4} + \frac{(\Gamma_3 + \Gamma_1)^2}{s_2^4} + \frac{(\Gamma_1 + \Gamma_2)^2}{s_3^4} \\ &+ 2 \left[\frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2} \right]. \end{aligned}$$

Thus, from Eqs. (25a) and (25b),

$$\begin{aligned} (2\pi\Omega)^2 - \left(\frac{\pi}{\tau}\right)^2 &= \frac{1}{2} \left[\frac{(\Gamma_2 + \Gamma_3)^2}{s_1^4} + \frac{(\Gamma_3 + \Gamma_1)^2}{s_2^4} + \frac{(\Gamma_1 + \Gamma_2)^2}{s_3^4} \right]. \quad (25c) \end{aligned}$$

Subtracting Eq. (25c) from Eq. (25b), we obtain

$$\begin{aligned} \left(\frac{\pi}{\tau}\right)^2 &= \frac{1}{2} \left[\frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2} \right] \\ &- \frac{1}{4} \left[\frac{(\Gamma_2 + \Gamma_3)^2}{s_1^4} + \frac{(\Gamma_3 + \Gamma_1)^2}{s_2^4} + \frac{(\Gamma_1 + \Gamma_2)^2}{s_3^4} \right]. \quad (25d) \end{aligned}$$

The relations (25a) and (25d) cannot accommodate a completely stationary configuration as is seen, for example, by considering Eq. (25b) or Eq. (25c) with 0 on the left hand side. Also note that while Ω is always positive according to Eq. (25a), τ can have either sign, as we have seen, but Eq. (25d) does not give the sign of τ . In fact, for given circulations a vortex triangle and its mirror image would have opposite values of τ , since one would expand while the other would collapse.

C. Relative equilibria

For an equilateral vortex triangle, $s_1 = s_2 = s_3 = s$, Eq. (25a) reproduces the known expression for its angular velocity of rotation, $\Omega = \gamma_1/2\pi s^2$. In this case we also get $\tau = \infty$ from Eq. (25d).

For a collinear relative equilibrium configuration, with the vortices along the x -axis, the quantities $\zeta_1, \zeta_2, \zeta_3$ are all real. In Ref. 16 we denoted the corresponding variables

ξ_1, ξ_2, ξ_3 and we shall use that notation here also. In the formulae above we now have $s_1^2 = \xi_1^2$, $s_2^2 = \xi_2^2$, and $s_3^2 = \xi_3^2$.

The stationary equilibrium is given by $\Gamma_1 \xi_1 = \Gamma_2 \xi_2 = \Gamma_3 \xi_3$. It is not part of the present family of solutions in the sense that it does not arise as a limiting case of self-similarly expanding or collapsing solutions. It was shown to be linearly unstable.¹⁶

We have just shown, as a special case of the preceding analysis, that the two collinear relative equilibria corresponding to points E_1 and E_3 in Fig. 1 both rotate with angular frequency

$$\Omega = \frac{1}{4\pi} \left[\frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} + \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} \right]. \quad (26)$$

This expression appears to differ from Eq. (67) of Ref. 16. To reconcile our current and former results, we note some useful consequences of the relation $L=0$, viz.,

$$\Gamma_1 \Gamma_2 \xi_3^2 + \Gamma_2 \Gamma_3 \xi_1^2 + \Gamma_3 \Gamma_1 \xi_2^2 = 0, \quad (27)$$

for collinear configurations. First, using $\xi_1 + \xi_2 + \xi_3 = 0$, we rewrite this equation as

$$\Gamma_1 \Gamma_2 (\xi_1 + \xi_2)^2 + \Gamma_2 \Gamma_3 (\xi_2 + \xi_3)^2 + \Gamma_3 \Gamma_1 (\xi_3 + \xi_1)^2 = 0.$$

Expanding the three squares and using $\gamma_2=0$, we see that the terms containing a square cancel one another by Eq. (27). We are left with the mixed terms

$$\Gamma_1 \Gamma_2 \xi_1 \xi_2 + \Gamma_2 \Gamma_3 \xi_2 \xi_3 + \Gamma_3 \Gamma_1 \xi_3 \xi_1 = 0. \quad (28a)$$

This is our first useful relation. We may write it in the equally useful form

$$\frac{\Gamma_1 \Gamma_2}{\xi_3} + \frac{\Gamma_2 \Gamma_3}{\xi_1} + \frac{\Gamma_3 \Gamma_1}{\xi_2} = 0. \quad (28b)$$

Squaring, we get

$$\begin{aligned} & \frac{\Gamma_1^2 \Gamma_2^2}{\xi_3^2} + \frac{\Gamma_2^2 \Gamma_3^2}{\xi_1^2} + \frac{\Gamma_3^2 \Gamma_1^2}{\xi_2^2} + 2\Gamma_1 \Gamma_2 \Gamma_3 \left[\frac{\Gamma_1}{\xi_2 \xi_3} + \frac{\Gamma_2}{\xi_3 \xi_1} + \frac{\Gamma_3}{\xi_1 \xi_2} \right] \\ &= 0. \end{aligned}$$

Dividing through by $\Gamma_1 \Gamma_2 \Gamma_3$, and using $\gamma_2=0$, now gives the second useful relation

$$\frac{\Gamma_1 + \Gamma_2}{\xi_3^2} + \frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} = 2 \left[\frac{\Gamma_1}{\xi_2 \xi_3} + \frac{\Gamma_2}{\xi_3 \xi_1} + \frac{\Gamma_3}{\xi_1 \xi_2} \right]. \quad (28c)$$

This identity reconciles Eq. (25a) with our earlier result for the angular frequency of rotation of the two collinear, rotating, relative equilibria.¹⁶ We repeat that Eq. (25a) immediately shows that the angular frequency is positive, i.e., that the two collinear relative equilibria rotate counterclockwise (with the same frequency).

We also note that marginal stability of these two relative equilibria now follows, since according our earlier analysis in Ref. 16, in particular Eq. (61a) of that paper, the sum of Ω and the third eigenvalue, Λ , is twice the right hand side of Eq. (25a). Thus, $\Omega = \Lambda$ and we have marginal stability of both rotating, collinear relative equilibria. This derivation is simpler than the one given previously.¹⁶

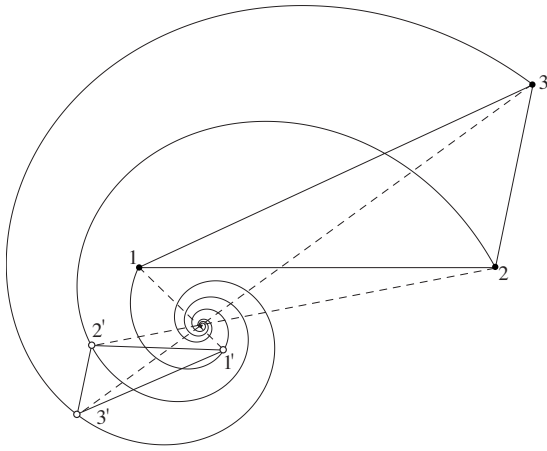


FIG. 2. Self-similar collapse of three vortices with $\Gamma_1:\Gamma_2:\Gamma_3=15:10:(-6)$. After Fig. 6 of Ref. 4. Labeling of the vortices conforms to the convention $\Gamma_3 < 0 < \Gamma_2 \leq \Gamma_1$. Initial condition 123; later positions 1'2'3'.

D. Shape of the trajectories

We obtain the shape of a vortex trajectory by eliminating time between the expression for the radial variable,

$$\rho_\alpha = |z_\alpha(0)| \sqrt{1 - \frac{t}{\tau}}, \quad (29a)$$

and the expression for the polar angle,

$$\varphi_\alpha = -\Omega\tau \log\left(1 - \frac{t}{\tau}\right). \quad (29b)$$

This gives

$$\rho_\alpha = \rho_\alpha(0) \exp\left(-\frac{\varphi_\alpha}{2\Omega\tau}\right), \quad (29c)$$

which we recognize as the equation for a *logarithmic spiral*. The trajectories of the vortices in a self-similar motion are similar logarithmic spirals winding into the center of vorticity. The trajectory of any vortex can, thus, be obtained by taking the trajectory of vortex 1 and rotating it through a suitable angle. We emphasize that this argument concerning the shape of trajectories derives directly from the scaling form of the vortex positions, Eq. (10a) and so applies to self-similar motions in the N -vortex problem. Figure 2, similar to an illustration in Gröbli's thesis,⁴ shows a case of self-similar collapse. Kimura¹⁷ gives two illustrations, one for three vortices and one for four.

Self-similar motions with more than three vortices have not been studied in much detail. Novikov and Sedov⁸ gave limited results for $N=4$. See also O'Neil¹⁸ for a recent study. The necessary conditions $\gamma_2=0$ and $L=0$ apply for all N but are no longer sufficient to determine the configurations that evolve self-similarly when $N \geq 4$. One then has to return to Eq. (13) which is similar to the equations for relative equilibria. Based on our experiences with the latter problem, we conjecture that configurations leading to self-similar motions exist for all $N \geq 3$. In particular, collapse to a point in a finite time remains a possibility for arbitrarily large N .

III. LINEAR STABILITY OF SELF-SIMILAR MOTIONS

The linear stability problem for self-similar motions may be approached in much the same way as we did for relative equilibria in Ref. 16. The base solution is given by Eq. (10a). Let us designate it $z_\alpha^{(0)}(t) = \lambda(t) z_\alpha^{(0)}(0)$. We take the perturbed solution in the form $z_\alpha(t) = z_\alpha^{(0)}(t) + \eta_\alpha(t) \lambda(t)$, where the η_α are infinitesimal. Factoring out $\lambda(t)$ is a matter of convenience. Now we obtain from Eq. (1) by expanding to linear order:

$$\begin{aligned} \dot{\bar{z}}_\alpha &= \dot{\bar{\lambda}} [\bar{z}_\alpha(0) + \bar{\eta}_\alpha] + \bar{\lambda} \dot{\bar{\eta}}_\alpha \\ &= \frac{1}{2\pi i} \sum_{\beta=1}^N \frac{\Gamma_\beta}{\lambda [z_\alpha(0) - z_\beta(0) + \eta_\alpha - \eta_\beta]} \\ &= \frac{1}{2\pi i \lambda} \sum_{\beta=1}^N \left\{ \frac{\Gamma_\beta}{z_\alpha(0) - z_\beta(0)} - \frac{\Gamma_\beta (\eta_\alpha - \eta_\beta)}{[z_\alpha(0) - z_\beta(0)]^2} \right\}. \end{aligned}$$

In an easily understood matrix-vector notation¹⁶ we have

$$|\lambda|^2 \dot{\bar{\eta}} + \lambda \dot{\bar{\lambda}} \bar{\eta} = iA \bar{\eta}, \quad (30a)$$

where A is given in terms of the variables $\zeta_1, \zeta_2, \zeta_3$, all evaluated at $t=0$, by

$$A = \frac{1}{2\pi} \begin{bmatrix} \frac{\Gamma_2}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_2^2} & -\frac{\Gamma_2}{\zeta_3^2} & -\frac{\Gamma_3}{\zeta_2^2} \\ -\frac{\Gamma_1}{\zeta_3^2} & \frac{\Gamma_1}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_1^2} & -\frac{\Gamma_3}{\zeta_1^2} \\ -\frac{\Gamma_1}{\zeta_2^2} & -\frac{\Gamma_2}{\zeta_1^2} & \frac{\Gamma_1}{\zeta_2^2} + \frac{\Gamma_2}{\zeta_1^2} \end{bmatrix}. \quad (30b)$$

As in the linear stability analysis for relative equilibria,¹⁶ where we wrote the infinitesimal perturbation as $\eta_\alpha(t) e^{i\Omega t}$, the benefit of factoring out $\lambda(t)$ is that the matrix (30b) becomes independent of time. In Eq. (30a) we have $|\lambda|^2 = 1 - (t/\tau)$, and $\lambda \dot{\bar{\lambda}} = -(1/2\tau) - i\Omega$. Thus, we get

$$\left(1 - \frac{t}{\tau}\right) \dot{\bar{\eta}} - \frac{1}{2\tau} \bar{\eta} = i(\Omega \bar{\eta} + A \bar{\eta}). \quad (31a)$$

This is the counterpart of Eq. (50a) in Ref. 16.

Setting

$$\chi(t) = \sqrt{1 - \frac{t}{\tau}} \bar{\eta}(t), \quad (31b)$$

we get (for $t < \tau$ when τ is positive)

$$\begin{aligned} \dot{\chi} &= -\frac{1}{2\tau \sqrt{1 - \frac{t}{\tau}}} \bar{\eta} + \sqrt{1 - \frac{t}{\tau}} \dot{\bar{\eta}} \\ &= \frac{1}{\sqrt{1 - \frac{t}{\tau}}} \left[\left(1 - \frac{t}{\tau}\right) \dot{\bar{\eta}} - \frac{1}{2\tau} \bar{\eta} \right]. \end{aligned}$$

Thus, Eq. (31a) becomes

$$\left(1 - \frac{t}{\tau}\right) \dot{\bar{\chi}} = i(\Omega \bar{\chi} + A \chi). \quad (31c)$$

Here we change independent variable from t to s by

$$s = -\tau \log\left(1 - \frac{t}{\tau}\right), \quad ds = \frac{dt}{1 - \frac{t}{\tau}}, \quad \left(1 - \frac{t}{\tau}\right) \frac{d}{dt} = \frac{d}{ds}.$$

Then Eq. (31c) becomes

$$\frac{d\bar{\chi}}{ds} = i(\Omega \bar{\chi} + A \chi). \quad (31d)$$

We differentiate this equation once more with respect to s and use Eq. (31d) in the result to obtain a second order ODE:

$$\frac{d^2 \chi}{ds^2} = (-\Omega^2 + \bar{A}A) \chi. \quad (31e)$$

This is the counterpart of Eq. (51) in Ref. 16.

We see by inspection that $e = (1, 1, 1)$ is an eigenvector of A with eigenvalue 0. Also, from Eq. (13) the vector $z^{(0)}$ satisfies

$$A z^{(0)} = \left(\Omega - \frac{i}{2\tau}\right) \bar{z}^{(0)},$$

such that

$$\begin{aligned} \bar{A}A z^{(0)} &= \left(\Omega - \frac{i}{2\tau}\right) \bar{A} \bar{z}^{(0)} \\ &= \left(\Omega - \frac{i}{2\tau}\right) \left(\Omega + \frac{i}{2\tau}\right) z^{(0)} = \left[\Omega^2 + \left(\frac{1}{2\tau}\right)^2\right] z^{(0)}. \end{aligned}$$

Thus, we know two eigenvalues, 0 and $\Omega^2 + (1/2\tau)^2$, of $\bar{A}A$. Since the trace of $\bar{A}A$ is real, it follows that the third eigenvalue, Λ , is also real. We calculate it in the next subsection.

The eigenmode e leads to χ evolving according to

$$\frac{d^2 \chi}{ds^2} = -\Omega^2 \chi,$$

$$\chi(s) = \chi(0) \exp(\pm i\Omega s),$$

$$\chi(t) = \left(1 - \frac{t}{\tau}\right)^{\mp i\Omega\tau} \chi(0).$$

Thus, the amplitude of χ remains constant in time. Since $\eta(t) = \chi(t) [1 - (t/\tau)]^{-1/2}$, we see that if $\tau < 0$, corresponding to self-similar expansion, the amplitude of η will decrease with time. On the other hand, if $\tau > 0$, corresponding to self-similar collapse, the amplitude of η will diverge as $t \rightarrow \tau$.

A similar conclusion is reached for the eigenmode $z^{(0)}$. A perturbation proportional to $z^{(0)}$ leads to χ evolving according to

$$\frac{d^2 \chi}{ds^2} = \left(\frac{1}{2\tau}\right)^2 \chi,$$

$$\chi(s) = \chi(0) \exp\left(\pm \frac{s}{2\tau}\right),$$

$$\chi(t) = \left(1 - \frac{t}{\tau}\right)^{\mp 1/2} \chi(0),$$

in other words to η evolving as

$$\eta(t) = \eta(0), \quad \text{or} \quad \eta(t) = \frac{\eta(0)}{1 - \frac{t}{\tau}}.$$

The first of these is marginally stable (it does not grow beyond the initial perturbation). The second will either grow or decay depending on the sign of τ . For self-similar expansion, $\tau < 0$, the perturbation decays as $t \rightarrow \infty$. For collapse, $\tau > 0$, the perturbation grows algebraically as $t \rightarrow \tau$.

We now consider to what extent these formal results signal stability or instability of the self-similar motions. Both perturbations considered are very simple and what is happening in the dynamics is quite transparent. The perturbation e corresponds to a rigid displacement of the original fluid triangle. The initial vortex separations are unchanged by such a perturbation. Thus, as far as Eq. (2) is concerned, the dynamics is unchanged. At the level of Eq. (10a) we have a base solution, $z_\alpha^{(0)}(t)$, with initial positions $z_\alpha^{(0)}(0)$. The perturbed solution, $z_\alpha(t)$, evolves from initial positions $z_\alpha(0) = z_\alpha^{(0)}(0) + a$, where a is an arbitrary displacement that is the same for all three vortices. The right hand sides in the dynamical equation (1) are unchanged by a common displacement of the vortices. Hence, $z_\alpha(t)$ will evolve as $z_\alpha^{(0)}(t) + a$. In the expanding self-similar motion, a becomes negligible compared to $z_\alpha^{(0)}(t)$ as $t \rightarrow \infty$, which implies stability in the linear analysis. For the collapse motion, on the other hand, $z_\alpha^{(0)}(t) \rightarrow 0$ as $t \rightarrow \tau$, whereas $z_\alpha(t) \rightarrow a$, and this finite deviation shows up as an instability in the linear analysis. The instability is algebraic rather than exponential precisely because the variation of positions in the collapsing motion is algebraic. The instability simply measures the deviation from collapse at the original center of vorticity (the origin of coordinates) relative to collapse at the displaced center of vorticity.

For the perturbation $z^{(0)}$ we are considering a change from the base solution, $z_\alpha^{(0)}(t)$, to $z_\alpha(t) = (1 + \epsilon) z_\alpha^{(0)}(t)$, where ϵ is a parameter that may be complex and that, in the context of linear stability, we would consider to be infinitesimal. In such a perturbation the initial vortex triangle is similar to the triangle formed by the base solution. Hence, the self-similar motion proceeds as for the base solution, although with τ multiplied and Ω divided by $|1 + \epsilon|^2$. In particular, the product $\Omega\tau$, which appears in the exponent of $\lambda(t)$, is invariant under this perturbation (even when the perturbation is not infinitesimal). We are, thus, comparing the base solution,

$$z_\alpha^{(0)}(t) = z_\alpha^{(0)}(0) \left(1 - \frac{t}{\tau}\right)^{(1/2) - i\Omega\tau},$$

to the perturbed solution,

$$z_\alpha(t) = z_\alpha^{(0)}(0) \left(1 - \frac{t}{\tau'}\right)^{(1/2) - i\Omega\tau},$$

where $\tau' = |1 + \epsilon|^2 \tau$. In other words,

$$\begin{aligned}
z_\alpha(t) &= z_\alpha^{(0)} \left(t \frac{\tau}{\tau'} \right) \\
&= z_\alpha^{(0)}(0) \lambda \left(\frac{t}{|1 + \epsilon|^2} \right) \\
&\approx z_\alpha^{(0)}(t) - 2 \operatorname{Re}(\epsilon) z_\alpha^{(0)}(0) \dot{\lambda}(t),
\end{aligned}$$

to first order in ϵ . Here

$$\dot{\lambda} = \frac{-\frac{1}{2\tau} + i\Omega}{\bar{\lambda}} = \frac{-\frac{1}{2\tau} + i\Omega}{|\lambda|^2} \lambda = \frac{-\frac{1}{2\tau} + i\Omega}{1 - \frac{t}{\tau}} \lambda,$$

so

$$z_\alpha(t) \approx z_\alpha^{(0)}(t) \left[1 + 2 \operatorname{Re}(\epsilon) \frac{\frac{1}{2\tau} - i\Omega}{1 - \frac{t}{\tau}} \right].$$

If $\tau < 0$, corresponding to a self-similarly expanding motion, the addition becomes ever more negligible as $t \rightarrow \infty$. On the other hand, if $\tau > 0$, that is, for a collapsing motion, it diverges as $t \rightarrow \tau$. Our previous conclusions, that the expanding motions are stable while the collapse motions are unstable, are recovered.

The self-similar motions exist in a subspace of the full configuration space for three vortices with circulations such that $\gamma_2 = 0$, viz., the subspace defined by the conditions in Eq. (12). If we introduce a perturbation that violates these conditions, the motion will clearly deviate from the self-similar solution which is, in this sense, unstable to such perturbations. However, violating Eq. (12) corresponds to changing the linear and angular impulse of the motion. We would generally deem such perturbations to be impermissible in a stability calculation since they represent forces or torques applied to the system by external agents, and not the intrinsic stability of the system due to fluctuations that do not add momentum, angular momentum, or mechanical energy to the system. A perturbation proportional to the eigenvector \mathbf{e} , i.e., $\delta z_\alpha = a$, violates the conditions in Eq. (12) since it changes the linear impulse by $\gamma_1 a$, and $\gamma_1 \neq 0$, when $\gamma_2 = 0$.

In general, a perturbation, δz_α , to a given “base” configuration, $z_\alpha^{(0)}$, changes the linear and angular impulse by

$$\delta(X + iY) = \delta \sum_{\alpha=1}^3 \Gamma_\alpha z_\alpha = \sum_{\alpha=1}^3 \Gamma_\alpha \delta z_\alpha, \quad (32a)$$

$$\delta I = \delta \sum_{\alpha=1}^3 \Gamma_\alpha |z_\alpha|^2 = \sum_{\alpha=1}^3 \Gamma_\alpha (z_\alpha^{(0)} \delta \bar{z}_\alpha + \delta z_\alpha \bar{z}_\alpha^{(0)}), \quad (32b)$$

respectively. The change in the value of the Hamiltonian is

$$\begin{aligned}
\delta H &= \sum_{\alpha=1}^N \left[\frac{\partial H}{\partial x_\alpha} \delta x_\alpha + \frac{\partial H}{\partial y_\alpha} \delta y_\alpha \right] \\
&= \sum_{\alpha=1}^N \Gamma_\alpha \left(\frac{dx_\alpha^{(0)}}{dt} \delta y_\alpha - \frac{dy_\alpha^{(0)}}{dt} \delta x_\alpha \right) = \sum_{\alpha=1}^N \Gamma_\alpha \operatorname{Im}(\dot{z}_\alpha^{(0)} \delta z_\alpha) \\
&= \operatorname{Im} \left[\left(-\frac{1}{2\tau} - i\Omega \right) \sum_{\alpha=1}^N \Gamma_\alpha \bar{z}_\alpha^{(0)} \delta z_\alpha \right]. \quad (32c)
\end{aligned}$$

For $\delta z_\alpha = a$ we see that $\delta(X + iY) = a\gamma_1$, as just stated, and $\delta I = \bar{a}(X + iY) + a(X - iY)$, which vanishes when $X = Y = 0$. We also see that δH is linear in X and Y , and so vanishes. This perturbation, then, would be disallowed since it implies a change in the linear impulse of the vortex system, although the angular impulse and the Hamiltonian are not changed.

For $\delta z_\alpha = \epsilon z_\alpha^{(0)}$, on the other hand, $\delta(X + iY)$ becomes proportional to $X + iY$, which vanishes for the base solution. Similarly, δI becomes proportional to I , which also vanishes for the base solution. Further, δH also becomes proportional to I , and so vanishes. This perturbation, then, would be allowed since it preserves the integrals of motion. Hence, from a partial analysis based on just two of the three eigenmodes we conclude that the self-similarly collapsing motions are unstable.

A. The third eigenvalue

In order to explore the problem further and, in particular, ascertain the linear stability of the self-similarly expanding motions, we seek the third eigenvalue, Λ , of $\bar{\mathbf{A}}\mathbf{A}$ and its corresponding eigenvector. The eigenvalue may be found by calculating the trace of $\bar{\mathbf{A}}\mathbf{A}$. The 11-element of the product matrix $\bar{\mathbf{A}}\mathbf{A}$ is

$$\begin{aligned}
&\left| \frac{\Gamma_2}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_2^2} \right|^2 + \frac{\Gamma_1 \Gamma_2}{s_3^4} + \frac{\Gamma_3 \Gamma_1}{s_2^4} \\
&= \frac{\Gamma_2(\Gamma_1 + \Gamma_2)}{s_3^4} + \frac{\Gamma_3(\Gamma_3 + \Gamma_1)}{s_2^4} + \frac{\Gamma_2 \Gamma_3}{s_2^4 s_3^4} (\zeta_2^2 \bar{\zeta}_3^2 + \bar{\zeta}_2^2 \zeta_3^2).
\end{aligned}$$

Permuting indices to find the other two diagonal elements of the product matrix, we obtain after a brief calculation

$$\begin{aligned}
\operatorname{Tr}(\bar{\mathbf{A}}\mathbf{A}) &= \left(\frac{1}{2\pi} \right)^2 \left[\frac{(\Gamma_1 + \Gamma_2)^2}{s_3^4} + \frac{(\Gamma_2 + \Gamma_3)^2}{s_1^4} + \frac{(\Gamma_3 + \Gamma_1)^2}{s_2^4} \right. \\
&\quad + \frac{\Gamma_1 \Gamma_2}{s_1^4 s_2^4} (\zeta_1^2 \bar{\zeta}_2^2 + \bar{\zeta}_1^2 \zeta_2^2) + \frac{\Gamma_2 \Gamma_3}{s_2^4 s_3^4} (\zeta_2^2 \bar{\zeta}_3^2 + \bar{\zeta}_2^2 \zeta_3^2) \\
&\quad \left. + \frac{\Gamma_3 \Gamma_1}{s_3^4 s_1^4} (\zeta_3^2 \bar{\zeta}_1^2 + \bar{\zeta}_3^2 \zeta_1^2) \right]. \quad (33)
\end{aligned}$$

Here, since $\zeta_1 + \zeta_2 + \zeta_3 = 0$,

$$\begin{aligned}
\zeta_1^2 \bar{\zeta}_2^2 + \bar{\zeta}_1^2 \zeta_2^2 &= (\zeta_1 \bar{\zeta}_2 + \bar{\zeta}_1 \zeta_2)^2 - 2s_1^2 s_2^2 \\
&= (|\zeta_1 + \zeta_2|^2 - s_1^2 - s_2^2)^2 - 2s_1^2 s_2^2 \\
&= (s_3^2 - s_1^2 - s_2^2)^2 - 2s_1^2 s_2^2 \\
&= s_1^4 + s_2^4 + s_3^4 - 2s_3^2(s_1^2 + s_2^2).
\end{aligned}$$

Hence, the sum of the last three terms in the expression for the trace, Eq. (33), is

$$\begin{aligned} & \left(\frac{\Gamma_1 \Gamma_2}{s_1^4 s_2^4} + \frac{\Gamma_2 \Gamma_3}{s_2^4 s_3^4} + \frac{\Gamma_3 \Gamma_1}{s_3^4 s_1^4} \right) (s_1^4 + s_2^4 + s_3^4) \\ & - 2 \left(\frac{\Gamma_1 \Gamma_2 s_3^2 (s_1^2 + s_2^2)}{s_1^4 s_2^4} + \frac{\Gamma_2 \Gamma_3 s_1^2 (s_2^2 + s_3^2)}{s_2^4 s_3^4} \right. \\ & \left. + \frac{\Gamma_3 \Gamma_1 s_2^2 (s_3^2 + s_1^2)}{s_3^4 s_1^4} \right). \end{aligned} \quad (34a)$$

In the last bracketed term of this expression we use $\Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 + \Gamma_1 \Gamma_2 s_3^2 = 0$, and then $\gamma_2 = 0$, to rewrite the numerators, e.g.,

$$\begin{aligned} \frac{\Gamma_1 \Gamma_2 s_3^2 (s_1^2 + s_2^2)}{s_1^4 s_2^4} &= - \frac{(\Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2) (s_1^2 + s_2^2)}{s_1^4 s_2^4} \\ &= \frac{\Gamma_1 \Gamma_2}{s_1^2 s_2^2} - \frac{\Gamma_3 \Gamma_1}{s_1^4} - \frac{\Gamma_2 \Gamma_3}{s_2^4}. \end{aligned}$$

Thus we obtain, upon applying $\gamma_2 = 0$ and $\Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 + \Gamma_1 \Gamma_2 s_3^2 = 0$ once more, that

$$\begin{aligned} & \frac{\Gamma_1 \Gamma_2 s_3^2 (s_1^2 + s_2^2)}{s_1^4 s_2^4} + \frac{\Gamma_2 \Gamma_3 s_1^2 (s_2^2 + s_3^2)}{s_2^4 s_3^4} + \frac{\Gamma_3 \Gamma_1 s_2^2 (s_3^2 + s_1^2)}{s_3^4 s_1^4} \\ &= \frac{\Gamma_2 \Gamma_3}{s_1^4} + \frac{\Gamma_3 \Gamma_1}{s_2^4} + \frac{\Gamma_1 \Gamma_2}{s_3^4}. \end{aligned} \quad (34b)$$

Using $\gamma_2 = 0$, the nine terms resulting from the product in the first term of Eq. (34a) may be written as

$$\frac{\Gamma_1 \Gamma_2 s_3^4}{s_1^4 s_2^4} + \frac{\Gamma_2 \Gamma_3 s_1^4}{s_2^4 s_3^4} + \frac{\Gamma_3 \Gamma_1 s_2^4}{s_3^4 s_1^4} - \left(\frac{\Gamma_1 \Gamma_2}{s_3^4} + \frac{\Gamma_2 \Gamma_3}{s_1^4} + \frac{\Gamma_3 \Gamma_1}{s_2^4} \right).$$

We may further transform the first three quantities in this expression as follows:

$$\begin{aligned} \frac{\Gamma_1 \Gamma_2 s_3^4}{s_1^4 s_2^4} &= \frac{(\Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2)^2}{\Gamma_1 \Gamma_2 s_1^4 s_2^4} \\ &= \frac{\Gamma_2^2 \Gamma_3^2}{\Gamma_1 s_1^4} + \frac{\Gamma_3^2 \Gamma_1^2}{\Gamma_2 s_2^4} + \frac{2 \Gamma_2 \Gamma_3}{s_1^2 s_2^2} \\ &= - \frac{(\Gamma_2 + \Gamma_3) \Gamma_3}{s_2^4} - \frac{(\Gamma_3 + \Gamma_1) \Gamma_1}{s_1^4} + \frac{2 \Gamma_2^2}{s_1^2 s_2^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\Gamma_2 \Gamma_3 s_1^4}{s_2^4 s_3^4} &= - \frac{(\Gamma_3 + \Gamma_1) \Gamma_1}{s_3^4} - \frac{(\Gamma_1 + \Gamma_2) \Gamma_2}{s_2^4} + \frac{2 \Gamma_1^2}{s_2^2 s_3^2}, \\ \frac{\Gamma_3 \Gamma_1 s_2^4}{s_3^4 s_1^4} &= - \frac{(\Gamma_1 + \Gamma_2) \Gamma_2}{s_1^4} - \frac{(\Gamma_2 + \Gamma_3) \Gamma_3}{s_3^4} + \frac{2 \Gamma_2^2}{s_3^2 s_1^2}. \end{aligned}$$

Adding these three equations we have

$$\begin{aligned} & \frac{\Gamma_1 \Gamma_2 s_3^4}{s_1^4 s_2^4} + \frac{\Gamma_2 \Gamma_3 s_1^4}{s_2^4 s_3^4} + \frac{\Gamma_3 \Gamma_1 s_2^4}{s_3^4 s_1^4} - \left(\frac{\Gamma_2 \Gamma_3}{s_1^4} + \frac{\Gamma_3 \Gamma_1}{s_2^4} + \frac{\Gamma_1 \Gamma_2}{s_3^4} \right) \\ &= - \left(\frac{\Gamma_2^2 + \Gamma_3^2}{s_1^4} + \frac{\Gamma_3^2 + \Gamma_1^2}{s_2^4} + \frac{\Gamma_1^2 + \Gamma_2^2}{s_3^4} \right) \\ &+ 2 \left(\frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2} \right). \end{aligned} \quad (34c)$$

Putting these partial results into the expression for the trace, Eq. (33), and using the obvious relation

$$\begin{aligned} & \frac{(\Gamma_2 + \Gamma_3)^2}{s_1^4} + \frac{(\Gamma_3 + \Gamma_1)^2}{s_2^4} + \frac{(\Gamma_1 + \Gamma_2)^2}{s_3^4} \\ &= \frac{\Gamma_2^2 + \Gamma_3^2}{s_1^4} + \frac{\Gamma_3^2 + \Gamma_1^2}{s_2^4} + \frac{\Gamma_1^2 + \Gamma_2^2}{s_3^4} \\ &+ 2 \left(\frac{\Gamma_2 \Gamma_3}{s_1^4} + \frac{\Gamma_3 \Gamma_1}{s_2^4} + \frac{\Gamma_1 \Gamma_2}{s_3^4} \right), \end{aligned}$$

we finally arrive at

$$\begin{aligned} \text{Tr}(\bar{A}A) &= 2 \left(\frac{1}{2\pi} \right)^2 \left[\frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2} \right] \\ &= 2 \left[\Omega^2 + \left(\frac{1}{2\tau} \right)^2 \right], \end{aligned} \quad (35)$$

where the last step follows from Eq. (25b). Since the trace is also the sum of the eigenvalues, $0 + \Omega^2 + (1/2\tau)^2 + \Lambda$, we conclude that

$$\Lambda = \Omega^2 + \left(\frac{1}{2\tau} \right)^2. \quad (36)$$

The temporal evolution of the third eigenmode, then, is identical to the evolution of the eigenmode $z^{(0)}$, that is, it implies linear stability for self-similar expansion, linear instability for self-similar collapse.

B. The third eigenmode

To determine the third eigenmode we return to the matrix A in Eq. (30b). It has the eigenvalue 0, and we denote its other two eigenvalues λ_1 and λ_2 . Consider the characteristic polynomial of the matrix $2\pi A$:

$$p_A(z) = \begin{vmatrix} \frac{\Gamma_2}{\xi_3^2} + \frac{\Gamma_3}{\xi_2^2} - z & -\frac{\Gamma_2}{\xi_3^2} & -\frac{\Gamma_3}{\xi_2^2} \\ -\frac{\Gamma_1}{\xi_3^2} & \frac{\Gamma_3}{\xi_1^2} + \frac{\Gamma_1}{\xi_3^2} - z & -\frac{\Gamma_3}{\xi_1^2} \\ -\frac{\Gamma_1}{\xi_2^2} & -\frac{\Gamma_2}{\xi_1^2} & \frac{\Gamma_1}{\xi_2^2} + \frac{\Gamma_2}{\xi_1^2} - z \end{vmatrix}. \quad (37a)$$

The coefficient of z^2 is given by the trace of $2\pi A$ and equals 2π times the sum of the eigenvalues of A . From this we conclude that

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi} \left(\frac{\Gamma_2 + \Gamma_3}{\zeta_1^2} + \frac{\Gamma_3 + \Gamma_1}{\zeta_2^2} + \frac{\Gamma_1 + \Gamma_2}{\zeta_3^2} \right). \quad (37b)$$

The coefficient of z in $p_A(z)$ is simply $-(2\pi)^2 \lambda_1 \lambda_2$ since one eigenvalue is 0. From this observation we get, after a brief calculation, that

$$\lambda_1 \lambda_2 = \frac{\gamma_1}{(2\pi)^2} \left(\frac{\Gamma_1}{\zeta_2 \zeta_3} + \frac{\Gamma_2}{\zeta_3 \zeta_1} + \frac{\Gamma_3}{\zeta_1 \zeta_2} \right). \quad (37c)$$

From the algebra of linear stability of the relative equilibria¹⁶ we are led to try if the vector,

$$\mathbf{v} = \left(\frac{\bar{\zeta}_1}{\Gamma_1}, \frac{\bar{\zeta}_2}{\Gamma_2}, \frac{\bar{\zeta}_3}{\Gamma_3} \right), \quad (38)$$

is an eigenvector of $\bar{\mathbf{A}}\mathbf{A}$. Hence, we proceed to calculate $\mathbf{A}\mathbf{v}$. In view of Eqs. (23a)–(23c), it is expedient to include a prefactor $2\pi\Omega - (i\pi/\tau)$. Thus, for the first component we consider

$$\begin{aligned} & \left(2\pi\Omega - \frac{i\pi}{\tau} \right) \left[\left(\frac{\Gamma_2}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_2^2} \right) \frac{\bar{\zeta}_1}{\Gamma_1} - \frac{\Gamma_2}{\zeta_3^2} \frac{\bar{\zeta}_2}{\Gamma_2} - \frac{\Gamma_3}{\zeta_2^2} \frac{\bar{\zeta}_3}{\Gamma_3} \right] \\ &= \frac{1}{\Gamma_1} \left(\frac{\Gamma_2}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_2^2} \right) \left(\frac{\Gamma_2 + \Gamma_3}{\zeta_1} + \frac{\Gamma_1 \zeta_1}{\zeta_2 \zeta_3} \right) \\ &\quad - \frac{1}{\zeta_3^2} \left(\frac{\Gamma_3 + \Gamma_1}{\zeta_2} + \frac{\Gamma_2 \zeta_2}{\zeta_3 \zeta_1} \right) - \frac{1}{\zeta_2^2} \left(\frac{\Gamma_1 + \Gamma_2}{\zeta_3} + \frac{\Gamma_3 \zeta_3}{\zeta_1 \zeta_2} \right). \end{aligned}$$

Straightforward simplification, using $\zeta_1 + \zeta_2 + \zeta_3 = 0$, gives the result

$$\begin{aligned} & \left(2\pi\Omega - \frac{i\pi}{\tau} \right) \left[\left(\frac{\Gamma_2}{\zeta_3^2} + \frac{\Gamma_3}{\zeta_2^2} \right) \frac{\bar{\zeta}_1}{\Gamma_1} - \frac{\Gamma_2}{\zeta_3^2} \frac{\bar{\zeta}_2}{\Gamma_2} - \frac{\Gamma_3}{\zeta_2^2} \frac{\bar{\zeta}_3}{\Gamma_3} \right] \\ &= \gamma_1 \left(\frac{\Gamma_2}{\zeta_1 \zeta_3^2} + \frac{\Gamma_3}{\zeta_1 \zeta_2^2} + \frac{\Gamma_1}{\zeta_2 \zeta_3^2} \right) \frac{\bar{\zeta}_1}{\Gamma_1}. \end{aligned}$$

Corresponding results hold for indices 2 and 3, and we conclude that

$$\left(\Omega - \frac{i}{2\tau} \right) \mathbf{A}\mathbf{v} = \frac{\gamma_1}{(2\pi)^2} \left(\frac{\Gamma_2}{\zeta_1 \zeta_3^2} + \frac{\Gamma_3}{\zeta_1 \zeta_2^2} + \frac{\Gamma_1}{\zeta_2 \zeta_3^2} \right) \bar{\mathbf{v}} = \lambda_1 \lambda_2 \bar{\mathbf{v}}.$$

Then,

$$\begin{aligned} & \left[\Omega^2 + \left(\frac{1}{2\tau} \right)^2 \right] \bar{\mathbf{A}}\mathbf{A}\mathbf{v} = \lambda_1 \lambda_2 \left(\Omega + \frac{i}{2\tau} \right) \bar{\mathbf{A}}\mathbf{v} \\ &= \lambda_1 \lambda_2 \overline{\left(\Omega - \frac{i}{2\tau} \right) \mathbf{A}\mathbf{v}} = |\lambda_1 \lambda_2|^2 \mathbf{v}. \end{aligned}$$

From our calculation in the previous subsection we know that $\Lambda = \Omega^2 + (1/2\tau)^2$ is the only nonzero eigenvalue of $\bar{\mathbf{A}}\mathbf{A}$, i.e., that $|\lambda_1 \lambda_2| = \Omega^2 + (1/2\tau)^2$. For a direct proof of this we transcribe

$$\begin{aligned} & \left| \frac{\Gamma_1}{\zeta_2 \zeta_3} + \frac{\Gamma_2}{\zeta_3 \zeta_1} + \frac{\Gamma_3}{\zeta_1 \zeta_2} \right|^2 \\ &= \frac{\Gamma_1^2}{s_2^4 s_3^4} + \frac{\Gamma_2^2}{s_3^4 s_1^4} + \frac{\Gamma_3^2}{s_1^4 s_2^4} + \frac{1}{s_1^4 s_2^4 s_3^4} [\Gamma_1 \Gamma_2 (\zeta_1^2 \bar{\zeta}_2^2 + \bar{\zeta}_1^2 \zeta_2^2) \\ &\quad + \Gamma_2 \Gamma_3 (\zeta_2^2 \bar{\zeta}_3^2 + \bar{\zeta}_2^2 \zeta_3^2) + \Gamma_3 \Gamma_1 (\zeta_3^2 \bar{\zeta}_1^2 + \bar{\zeta}_3^2 \zeta_1^2)] \end{aligned} \quad (39)$$

along the lines of the calculation of $\text{Tr}(\bar{\mathbf{A}}\mathbf{A})$. The terms in square brackets may be rewritten as

$$\Gamma_1 \Gamma_2 (\zeta_1^2 \bar{\zeta}_2^2 + \bar{\zeta}_1^2 \zeta_2^2) = \Gamma_1 \Gamma_2 [s_1^4 + s_2^4 + s_3^4 - 2s_3^2 (s_1^2 + s_2^2)],$$

$$\Gamma_2 \Gamma_3 (\zeta_2^2 \bar{\zeta}_3^2 + \bar{\zeta}_2^2 \zeta_3^2) = \Gamma_2 \Gamma_3 [s_1^4 + s_2^4 + s_3^4 - 2s_1^2 (s_2^2 + s_3^2)],$$

$$\Gamma_3 \Gamma_1 (\zeta_3^2 \bar{\zeta}_1^2 + \bar{\zeta}_3^2 \zeta_1^2) = \Gamma_3 \Gamma_1 [s_1^4 + s_2^4 + s_3^4 - 2s_2^2 (s_3^2 + s_1^2)].$$

Thus, since $\gamma_2 = 0$, the square bracket in Eq. (39) reduces to

$$\begin{aligned} & -2[\Gamma_1 \Gamma_2 s_3^2 (s_1^2 + s_2^2) + \Gamma_2 \Gamma_3 s_1^2 (s_2^2 + s_3^2) + \Gamma_3 \Gamma_1 s_2^2 (s_3^2 + s_1^2)] \\ &= 2(\Gamma_1 \Gamma_2 s_1^2 s_2^2 + \Gamma_2 \Gamma_3 s_2^2 s_3^2 + \Gamma_3 \Gamma_1 s_3^2 s_1^2) \\ &= 2s_1^2 s_2^2 s_3^2 \left(\frac{\Gamma_1 \Gamma_2}{s_3^2} + \frac{\Gamma_3 \Gamma_1}{s_2^2} + \frac{\Gamma_2 \Gamma_3}{s_1^2} \right). \end{aligned}$$

We now have

$$\left| \frac{\Gamma_1}{\zeta_2 \zeta_3} + \frac{\Gamma_2}{\zeta_3 \zeta_1} + \frac{\Gamma_3}{\zeta_1 \zeta_2} \right|^2 = \left(\frac{\Gamma_1}{s_2^2 s_3^2} + \frac{\Gamma_2}{s_3^2 s_1^2} + \frac{\Gamma_3}{s_1^2 s_2^2} \right)^2.$$

Multiplying by γ_1^2 , and taking the square root, we get

$$|\lambda_1 \lambda_2| = \frac{1}{(2\pi)^2} \left(\frac{\Gamma_1^2}{s_2^2 s_3^2} + \frac{\Gamma_2^2}{s_3^2 s_1^2} + \frac{\Gamma_3^2}{s_1^2 s_2^2} \right) = \Omega^2 + \left(\frac{1}{2\tau} \right)^2,$$

as desired. [Inside the parentheses we first get $\gamma_1 \Gamma_1 = \Gamma_1^2 + \Gamma_1(\Gamma_2 + \Gamma_3) = \Gamma_1^2 - \Gamma_2 \Gamma_3$, etc., but the sum of the three terms with mixed subscripts vanishes since $L=0$.] Finally,

$$\bar{\mathbf{A}}\mathbf{A}\mathbf{v} = \left[\Omega^2 + \left(\frac{1}{2\tau} \right)^2 \right] \mathbf{v}, \quad (40)$$

with the eigenvector, \mathbf{v} , given by Eq. (38).

Returning to the considerations in Eqs. (32a)–(32c), we see that perturbations proportional to \mathbf{v} conserve both linear and angular impulse and conserve the Hamiltonian:

$$\begin{aligned} & \delta \sum_{\alpha=1}^3 \Gamma_{\alpha} z_{\alpha} = \sum_{\alpha=1}^3 \Gamma_{\alpha} \delta z_{\alpha} \\ &= \epsilon \sum_{\alpha=1}^3 \Gamma_{\alpha} \frac{\bar{\zeta}_{\alpha}^{(0)}}{\Gamma_{\alpha}} = \epsilon (\bar{\zeta}_1^{(0)} + \bar{\zeta}_2^{(0)} + \bar{\zeta}_3^{(0)}) = 0 \end{aligned}$$

shows that the linear impulse is preserved.

$$\begin{aligned} & \delta \sum_{\alpha=1}^3 \Gamma_{\alpha} |z_{\alpha}^{(0)}|^2 = \sum_{\alpha=1}^3 \Gamma_{\alpha} (z_{\alpha}^{(0)} \delta \bar{z}_{\alpha} + \delta z_{\alpha} \bar{z}_{\alpha}^{(0)}) \\ &= \sum_{\alpha=1}^3 (\bar{\epsilon} z_{\alpha}^{(0)} \zeta_{\alpha}^{(0)} + \epsilon \bar{\zeta}_{\alpha}^{(0)} z_{\alpha}^{(0)}) = 0, \end{aligned}$$

where the last step follows from

$$\sum_{\alpha=1}^3 z_{\alpha}^{(0)} \zeta_{\alpha}^{(0)} = z_1^{(0)}(z_2^{(0)} - z_3^{(0)}) + z_2^{(0)}(z_3^{(0)} - z_1^{(0)}) \\ + z_3^{(0)}(z_1^{(0)} - z_2^{(0)}) = 0,$$

shows that the angular impulse is preserved. Finally, the change in the Hamiltonian is

$$\delta H = \text{Im} \left[\sum_{\alpha=1}^3 \Gamma_{\alpha} \left(-\frac{1}{2\tau} - i\Omega \right) \bar{z}_{\alpha}^{(0)} \delta z_{\alpha} \right] \\ = \text{Im} \left[\left(-\frac{1}{2\tau} - i\Omega \right) \epsilon \sum_{\alpha=1}^3 \bar{z}_{\alpha}^{(0)} \zeta_{\alpha}^{(0)} \right] = 0.$$

Thus, such perturbations are permissible according to our previous discussion. It is particularly interesting that the perturbation \mathbf{v} is not a simple affine transformation of the vortex positions as were the perturbations \mathbf{e} and $\mathbf{z}^{(0)}$.

In summary, within the subspace of initial conditions for which $L=0$ we have found a nontrivial, algebraically growing instability of the self-similar collapse motions. The self-similarly expanding motions are linearly stable. These conclusions were reached by Tavantzis and Ting⁹ and, using an approach more similar to this paper, by Kimura.¹⁰ However, the explicit expressions for the third eigenmode and the formulae for Ω and τ in terms of the configuration geometry appear to be new.

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